THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010G University Mathematics 2014-2015
Suggested Solution to Test 2

1. (a)

$$\begin{aligned}
& \lim_{x\to 1/2} \frac{e^x - e^{-x} - 2\sin x}{x^3} && (\frac{0}{0}) \\
= & \lim_{x\to 1/2} \frac{e^x + e^{-x} - 2\cos x}{3x^2} && (\frac{0}{0}) \\
= & \lim_{x\to 1/2} \frac{e^x - e^{-x} + 2\sin x}{6x} && (\frac{0}{0}) \\
= & \lim_{x\to 1/2} \frac{e^x + e^{-x} + 2\cos x}{6} \\
= & \frac{4}{6} \\
= & \frac{2}{3}
\end{aligned}$$

(b) Let $y = (e^x + x)^{1/x}$, so $\ln y = \dfrac{\ln(e^x + x)}{x}$. Then

$$\begin{aligned}
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\ln(e^x + x)}{x} && (\frac{0}{0}) \\
&= \lim_{x\to 0} \frac{e^x + 1}{e^x + x} \\
&= 2
\end{aligned}$$

Therefore, $\lim_{x\to 0} (\sin x)^{\tan x} = e^2$

2. Let $y = \tan^{-1} x$, that means $\tan y = x$ and so $\cos y = \dfrac{1}{\sqrt{1+x^2}}$. Then

$$\begin{aligned}
\tan y &= x \\
\sec^2 y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2 y \\
\frac{d}{dx} \tan^{-1} x &= \frac{1}{1+x^2}
\end{aligned}$$

3. We have

$$\begin{aligned}
f(x) &= \ln(1-x) \\
f'(x) &= \frac{-1}{1-x} \\
f''(x) &= \frac{-1}{(1-x)^2} \\
f'''(x) &= \frac{-2}{(x-1)^3}
\end{aligned}$$

Therefore, $f(0) = 0$, $f'(0) = -1$, $f''(0) = -1$ and $f'''(0) = -2$ and

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x + \frac{x^2}{2} + \frac{x^3}{3}$$

Then

$$\ln 0.99 = f(0.01) \approx P_3(0.01) = 0.01005033$$

4. (a) Let $f(t) = kt + (1-k) - t^k$, then $f'(t) = k - kt^{k-1} = k(1-t^{k-1})$. Note $0 < k < 1$, so $1-k > 0$.

$$\begin{aligned} f'(t) &> 0 \\ k(1-t^{k-1}) &> 0 \\ 1 &> t^{k-1} \\ t^{1-k} &> 1 \\ t &> 1 \end{aligned}$$

Similarly, $f'(t) < 0$ when $0 < t < 1$.

Therefore, $f(t) \geq f(1) = 0$ for all $t > 0$ and it follows that $kt + (1-k) \geq t^k$ for all $t > 0$.

(b) Since $r, s > 0$, $\frac{r}{s} > 0$. Using the result in (a),

$$\begin{aligned} k(\frac{r}{s}) + (1-k) &\geq (\frac{r}{s})^k \\ kr + (1-k)s &\geq r^k s^{1-k} \end{aligned}$$

5. (a) $f'(x) = (2x^2-1)e^{-x^2}$ and $f''(x) = -2x(2x^2-3)e^{-x^2} = -2x(\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})e^{-x^2}$.

(b) Note that $e^{-x^2} > 0$

(i) $f'(x) > 0$ when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

(ii) $f'(x) < 0$ when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$

(iii) $f''(x) > 0$ when $x < -\sqrt{\frac{3}{2}}$ or $0 < x < \sqrt{\frac{3}{2}}$

(iv) $f''(x) < 0$ when $-\sqrt{\frac{3}{2}} < x < 0$ or $x > \sqrt{\frac{3}{2}}$

(c) $f(x)$ has a local maximum point $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{-1/2})$ and a local minimum point $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}e^{-1/2})$.

(d) $f(x)$ has points of inflections $(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}e^{-3/2})$, $(0,0)$ and $(\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}e^{-3/2})$.

(e) Note $m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} -e^{-x^2} = 0$, and $c = \lim_{x\to\infty} f(x) - mx = \lim_{x\to\infty} -xe^{-x^2} = 0$.

Therefore, $f(x)$ has a horizontal asymptote $y = 0$.

(f) The graph of $f(x)$.